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Advances in Compressed Sensing & Sparse Signal Modelling

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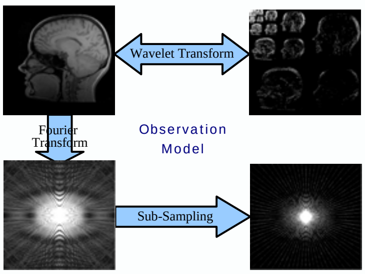
Compressed Sensing Overview

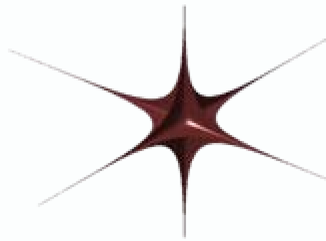
Compressed Sensing: an MRI example



The Image is sparse in the wavelet domain (top right). Measurements are taken in the Fourier domain (bottom left), but we would like to take as few measurements as possible (bottom right).

The Sparse Signal Model

Assume **y** is approximately sparse, i.e. **y** has many small elements.

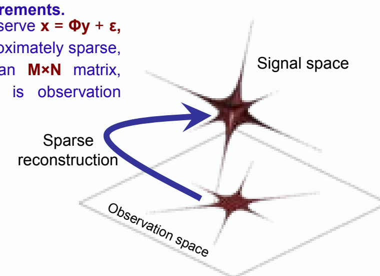


Three dimensional example where **y** is an element from an l_p ball.

The Challenge

Estimate the signal y from a few linear measurements.

We observe $\mathbf{x} = \boldsymbol{\Phi}\mathbf{y} + \boldsymbol{\varepsilon}$,
y approximately sparse,
$\boldsymbol{\Phi}$ is an **M×N** matrix,
and **ε** is observation noise



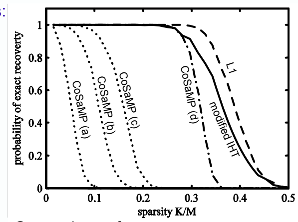
Q1: How many measurements do we need?
Q2: How do we take good measurements?
Q3: How do we reconstruct the original signal?

New Algorithms: provably fast & good

Finding the best K-sparse estimate $\hat{\mathbf{y}}$ (i.e. such that $\|\mathbf{x} - \boldsymbol{\Phi}\hat{\mathbf{y}}\|_2$ is minimal) is NP hard.

We have developed a range of extremely simple new greedy algorithms including *Iterated Hard Thresholding* (IHT) which we can prove has:

- Near-optimal reconstruction & approximation performance when $\boldsymbol{\Phi}$ has the restricted isometry property
- Bounded algorithmic complexity (number of iterations is proportional to logarithmic of SNR)



Comparison of sparse signal reconstruction with other state-of-the art techniques: L1 minimization and CoSAMP.

CoSaMP also has provably fast performance when using conjugate gradient updates in the inner loop. Here we show the performance for (a) 3 CG steps, (b) 6 CG steps, (c) 9 CG steps, and (d) full least squares.

The Iterative Hard Thresholding (IHT) algorithm

$$\mathbf{y}^{[n+1]} = H_K(\mathbf{y}^{[n]} + \boldsymbol{\Phi}^T(\mathbf{x} - \boldsymbol{\Phi}\,\mathbf{y}^{[n]}))$$

where H_K is the operator keeping the largest K elements. A modified version with an adaptive step-size further improves performance.

Furthermore IHT can be extended to reconstruct model-based signal representations (see "Future directions" box).

Better models: structured representations

The K-Sparse signal model is a union of subspaces. We have generalized CS theory to an arbitrary union of subspaces. This enables us to incorporate different structured models, including:

- Redundant representations (analysis or synthesis)
- Simultaneous sparse representations (e.g. multi-channel source separation)
- Multi-resolution sparse representations (e.g. tree based wavelet models)

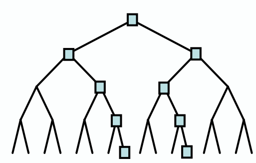
Classical compressed sensing shows that stable inverses exist when the number of samples (observation measurements), M satisfies:

$$M \geq \text{const.} \times K \log(N/K)$$

where the original signal is N dimensional.
For tree-restricted sparsity this reduces to:

$$M \geq \text{const.} \times K$$

Thus by restricting the number of subspaces we require many fewer samples.



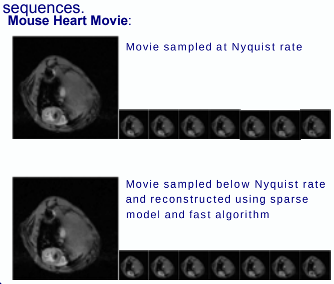
Example of sparse coefficients restricted to form a tree. This significantly reduces the number of possible subspaces

Some applications

Compressed Sensing in dynamic MRI

We are currently exploring better signal models, sampling strategies and reconstruction algorithms to enable rapid acquisition of dynamic MRI sequences.

Mouse Heart Movie:



Compressed Sensing in SAR

Under-sampling in Synthetic Aperture Radar can provide reduced data rates (for transmission) or to allow the radar to be intermittently used in other modes (interrupted SAR).

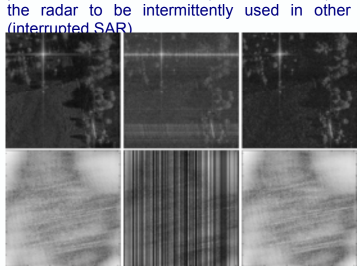


Image domain

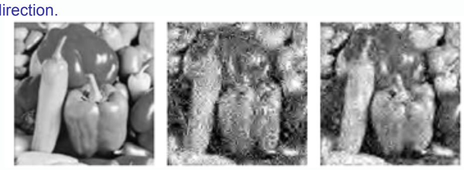
Fourier domain

fully sampled data | backprojection of sub-sampled data | sparse approx. of sub-sampled data

Future directions

Model-based Compressed Sensing

Our work on IHT + structured sparsity have recently been used by Rice University to perform Model-based Compressed Sensing. This enables, for example multi-resolution wavelet trees to be used to further reduce sub-sampling. We anticipate further advances in this direction.



original | sparse reconstruction | model based reconstruction

Images from: Baranuik et al. "Model-based Compressed Sensing", 2008

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- MOD Competition of Ideas award: "*Compressed Sensing SAR*", (**£323,736**), 2008-2011.
- EPSRC award: "*Sparse Representations in Signal Processing and Coding*" (**£265,442**) 2005-2009.